Math 742 H.W. #2

1. (Conway 3-4). Discuss the mapping properties of z^n and $z^{1/n}$ for $n \geq 2$.

Solution: Let $f(z) = z^n$. Then if $C_r = \{z \in \mathbb{C} : |z| = r\}$ is the circle of radius r, C_r is parametrized by $z(\theta) = re^{i\theta}$ $0 \leq \theta \leq 2\pi$ and $f(z(\theta)) = r^n e^{in\theta} \Rightarrow f(C_r) = C_{r^n}$, where C_r is "wrapped" n times onto C_{r^n}.

The function $g(z) = z^{1/n}$ has distinct branches. The principal branch is given by $g(z) = e^{1/n \log z}$ where $\log z = \ln|z| + i \operatorname{Arg} z$ $(-\pi < \operatorname{Arg} z \leq \pi)$. If $S_\theta = \{z \in \mathbb{C} : 0 \leq \operatorname{Arg} z \leq \theta\}$ is a sector of angle θ, $g(S_\theta) = S_{\theta/n}$. Also if $z = re^{i\theta}$, $g(z) = r^{1/n} e^{i\theta/n}$ so g carries a segment of C_r onto a segment of $C_{r^{1/n}}$.

2. (Conway 3-5) Find the fixed points of a dilation, translation, and inversion on $\mathbb{C}_\infty$.

Solution:

- Dilation: $S(z) = az \Rightarrow S(z) = z$ iff $(a-1)z = 0$ $(z \neq \infty)$

Hence, unless $a = 1$, the fixed points are $z = 0$ and $z = \infty$

- Translation: $S(z) = z + b \Rightarrow S(z) = z$ iff $z = \infty$. Thus

$z = \infty$ is the only fixed point.

- Inversion: $S(z) = 1/z \Rightarrow S(z) = z$ iff $z^2 = 1$. Thus

$z = \pm 1$ are the fixed points.

3. (Conway 3-6) Evaluate the following cross ratios:

(a) $(7+i, 1, 0, \infty)$

(b) $(2, i-1, 1, 1+i)$

(c) $(0, 1, i, -1)$

(d) $(i-1, \infty, 1+i, 0)$

Solution: Let $S(z) = \begin{pmatrix} z, & z_2, & z_3, & z_4 \\ & \downarrow & \downarrow & \downarrow \\ & 0 & 1 & \infty \end{pmatrix}$. Note that this definition is different from Conway's.

Then $S(z) = \dfrac{z - z_2}{z - z_4} \cdot \dfrac{z_3 - z_4}{z_3 - z_2}$ if $z_2, z_3, z_4 \in \mathbb{C}$.

$S(z) = \dfrac{z_3 - z_4}{z - z_4}$ if $z_2 = \infty$

$S(z) = \dfrac{z - z_2}{z - z_4}$ if $z_3 = \infty$

$S(z) = \dfrac{z - z_2}{z_3 - z_2}$ if $z_4 = \infty$

Hence

(a) $(7+i, 1, 0, \infty) = \dfrac{7+i-1}{0-1} = -(6+i) = -6-i$

(b) $(2, i-1, 1, 1+i) = \dfrac{2-(i-1)}{2-(i+1)} \cdot \dfrac{1-(1+i)}{1-(i-1)} = \dfrac{3-i}{1-i} \cdot \dfrac{-i}{2-i} =$

$= \dfrac{-1-3i}{1-3i} = -\dfrac{1+3i}{1-3i} = -\dfrac{(1+3i)(1+3i)}{10} = -\dfrac{(-8+6i)}{10}$

$= \frac{1}{5}(4-3i)$

(c) $(0, 1, i, -1) = \dfrac{0-1}{0+1} \cdot \dfrac{i+1}{i-1} = -1 \dfrac{i+1}{i-1} = -(i+1)(-i-1) \cdot \frac{1}{2}$

$= \frac{1}{2}(1+i)^2 = \frac{1}{2}(2i) = i$

(d) $(i-1, \infty, 1+i, 0) = \frac{1+i-0}{i-1-0} = \frac{1+i}{-1+i} = (1+i)(-1-i)\cdot\frac{1}{2} =$
$= -\frac{1}{2}(1+i)^2 = -\frac{1}{2}(2i) = -i$

4. (Conway 3-7) If $Tz = \frac{az+b}{cz+d}$, find z_2, z_3, z_4 (in terms of a,b,c,d) such that $Tz = (z, z_2, z_3, z_4)$.

Solution: Observe that $T^{-1}z = \frac{dz-b}{-cz+a}$ and $T^{-1}(0) = \frac{-b}{a}$, $T^{-1}(1) = \frac{d-b}{a-c}$, $T^{-1}(\infty) = -\frac{d}{c}$. Therefore

$Tz = \left(z, -\frac{b}{a}, \frac{d-b}{a-c}, -\frac{d}{c}\right).$

5. (Conway 3-8). If $Tz = \frac{az+b}{cz+d}$, show that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ iff we can choose a,b,c,d to be real numbers.

Solution: If a,b,c,d are all in $\mathbb{R}_\infty$, $T(0), T(1), T(\infty) \in \mathbb{R}_\infty$ $\Rightarrow T(\mathbb{R}_\infty) \subseteq \mathbb{R}_\infty$ (Hence $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$) because Möbius maps carry circles to circles.
On the other hand, if $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$, $T^{-1}(\mathbb{R}_\infty) = \mathbb{R}_\infty$.
Therefore $Tz = (z, T^{-1}(0), T^{-1}(1), T^{-1}(\infty)) =$
$= \left(z, -\frac{b}{a}, \frac{d-b}{a-c}, -\frac{d}{c}\right) = (z, \alpha, \beta, \delta, \lambda)$ is a representation of T, where $\alpha, \beta, \delta, \lambda \in \mathbb{R}_\infty$.

6. (Conway 3-9) If $Tz = \frac{az+b}{cz+d}$, find necessary and sufficient conditions such that $T(S^1) = S^1$.

Solution: Let $\varphi(z) = \frac{z-i}{z+i}$. Then $\varphi(\mathbb{R}_\infty) = S^1$. Hence $\varphi^{-1}(z) = \frac{iz+i}{-z+1}$ carries S^1 onto $\mathbb{R}_\infty$.

Define the möbius map S by $S = \varphi^{-1} \circ T \circ \varphi$. Clearly $T(S^1) = S^1$ iff $S(\mathbb{R}_\infty) = \mathbb{R}_\infty$. By the previous exercise, this can happen iff we can pick $\alpha, \beta, \lambda, \delta \in \mathbb{R}_\infty$ such that $Sz = \dfrac{\alpha z + \beta}{\lambda z + \delta}$.

Thus $$\frac{az+b}{cz+d} = Tz = \varphi S \varphi^{-1} z = \varphi\left(\frac{\alpha\left(\frac{iz+i}{-z+1}\right)+\beta}{\lambda\left(\frac{iz+i}{-z+1}\right)+\delta}\right) =$$

$$= \varphi\left(\frac{i\alpha z + i\alpha - \beta z + \beta}{i\lambda z + i\lambda - \delta z + \delta}\right) = \varphi\left(\frac{(i\alpha-\beta)z + (i\alpha+\beta)}{(i\lambda-\delta)z + (i\lambda+\delta)}\right) =$$

$$= \frac{\dfrac{(i\alpha-\beta)z+(i\alpha+\beta)}{(i\lambda-\delta)z+(i\lambda+\delta)} - i}{\dfrac{(i\alpha-\beta)z+(i\alpha+\beta)}{(i\lambda-\delta)z+(i\lambda+\delta)} + i} = \frac{\left[(\lambda-\beta)+i(\alpha+\delta)\right]z + \left[(\lambda+\beta)+i(\alpha-\delta)\right]}{-\left[(\lambda+\beta)-i(\alpha-\delta)\right]z - \left[(\lambda-\beta)-i(\alpha+\delta)\right]}$$

Thus $d = -\bar{a}$ and $c = -\bar{b}$. It follows that $T(S^1) = S^1$ iff $Tz = \dfrac{az+b}{-\bar{b}z - \bar{a}} = -\dfrac{az+b}{\bar{b}z+\bar{a}}$

Remark: Since multiplication by any $e^{i\theta}$ carries S^1 onto S^1, we can also write $Tz = \dfrac{az+b}{\bar{b}z+\bar{a}}$ (multiplication by $e^{i\pi} = -1$) or $Tz = e^{i\theta}\dfrac{az+b}{\bar{b}z+\bar{a}}$.

7. (Conway 3-10) Consider the interior of the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Find all Möbius transformations T such that $T(\mathbb{D}) = \mathbb{D}$.

Solution: Since Möbius functions are continuous on $\mathbb{C}_\infty$, it is clear that $|Tz| \mapsto 1$ as $z \mapsto z_0 \in S^1 = \partial\mathbb{D}$ from within $\mathbb{D}$.

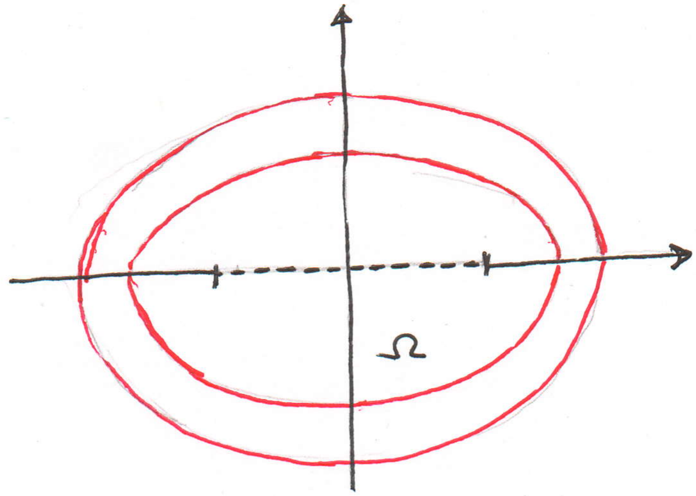
Hence $T(S') = S'$. By the previous problem, this means that $Tz = e^{i\theta} \frac{az+b}{\bar{b}z+\bar{a}}$. Furthermore, since $\mathbb{D}$ is connected, it is enough to insure that $T(0) = \frac{b}{\bar{a}} e^{i\theta} \in \mathbb{D} \iff \frac{b}{\bar{a}} \in \mathbb{D}$.

Remark: By adjusting the coefficients, it is possible to write $Tz = e^{i\theta} \frac{\omega - z}{1-\bar{\omega}z}$.

8. (Conway 3-13) Give a discussion of the mapping

$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Solution: let $\mathbb{D} = \{z : |z|<1\}$ and $\mathbb{D}^* = \{z : |z|>1\} \cup \{\infty\}$. Any $z \in \mathbb{D}^*$ has a polar representation of the form $z = re^{i\theta}$, $r>1$ and $\theta \in [0, 2\pi)$. Writing $f(re^{i\theta}) = \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right) = \left(\left(r+\frac{1}{r}\right)\cos\theta, \left(r-\frac{1}{r}\right)\sin\theta\right)$ shows that f carries the circle $|z|=r$ onto the ellipse $\frac{x^2}{(r+\frac{1}{r})^2} + \frac{y^2}{(r-\frac{1}{r})^2} = 1$ with semi-major axis length $r+\frac{1}{r}$ and semi-minor axis of length $r-\frac{1}{r}$. Furthermore, if $f(z_1) = f(z_2)$ where $z_k = r_k e^{i\theta_k}$, $r_k > 1$, $\theta_k \in [0, 2\pi)$, it follows that $r_1 = r_2$ for otherwise (if, say, $r_1 < r_2$) $|f(z_2) - f(z_1)| \geq \left(r_2 - \frac{1}{r_2}\right) - \left(r_1 - \frac{1}{r_1}\right) > 0$. Hence it instantly follows that $\theta_1 = \theta_2 \Rightarrow z_1 = z_2$. In particular f is injective on $\mathbb{D}^*$. If $r \mapsto \infty$, $f(re^{i\theta}) \mapsto \infty$ and if $r \mapsto 1^+$, $f(re^{i\theta}) \mapsto \cos\theta$. Hence f maps $\mathbb{D}^*$ bijectively onto $\Omega = \mathbb{C}_\infty - [-1,1]$. A similar analysis shows that $f : \mathbb{D} \mapsto \Omega$ is a conformal bijection.

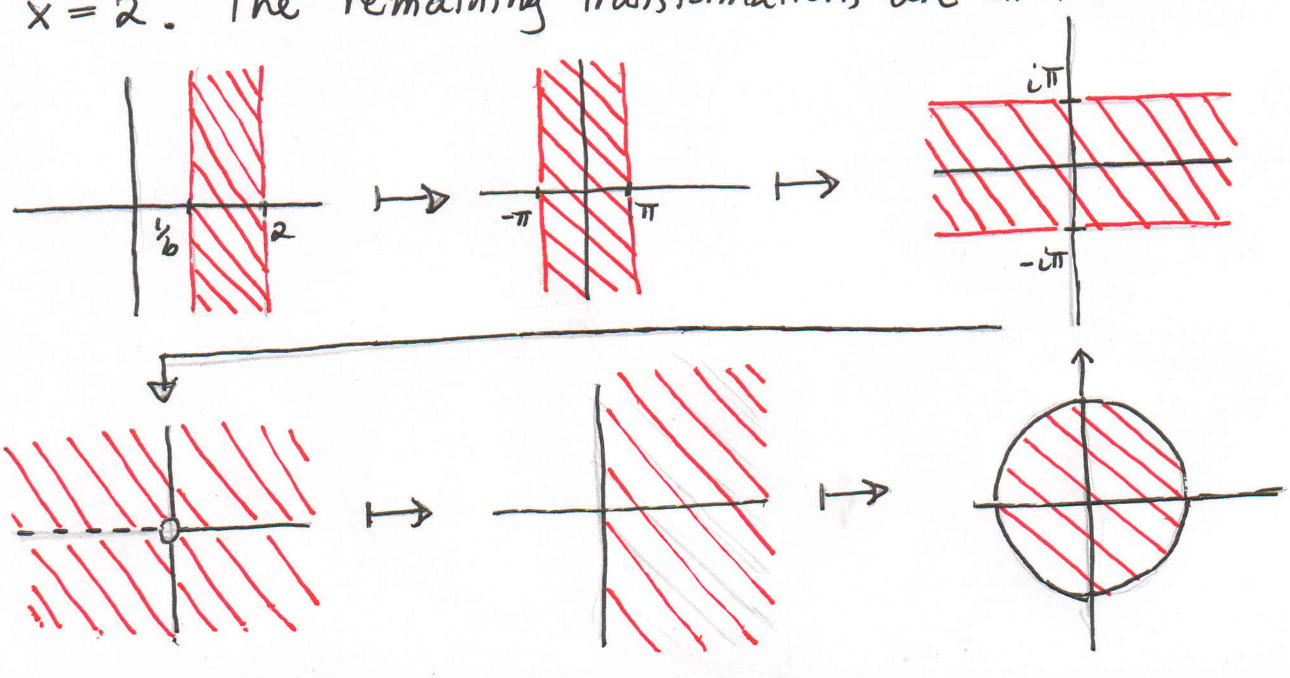


9. (Conway 3-14) Suppose that one circle is contained inside another and that they are tangent at the point a. Let G be the region between the two circles and map G conformally onto the open unit disk $\mathbb{D}$.

Solution: By scaling and rotating appropriately (and by shifting) we may assume that $a = 0$ and that the circles are

$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$ and $(x - b)^2 + y^2 = b^2$ where $b > \frac{1}{2}$.

The map $z \mapsto z^{-1}$ carries these circles onto the lines $x = \frac{1}{b}$ and $x = 2$. The remaining transformations are illustrated below:



10. (Conway 3-18) Let $-\infty < a < b < \infty$ and put $Mz = \frac{z-ia}{z-ib}$. Define the lines $L_1 = \{z : \operatorname{Im} z = b\}$, $L_2 = \{z : \operatorname{Im} z = a\}$ and $L_3 = \{z : \operatorname{Re} z = 0\}$. Determine which of the regions A, B, C, D, E, F in Figure 1 are mapped by M onto the regions U, V, W, X, Y, Z in Figure 2.

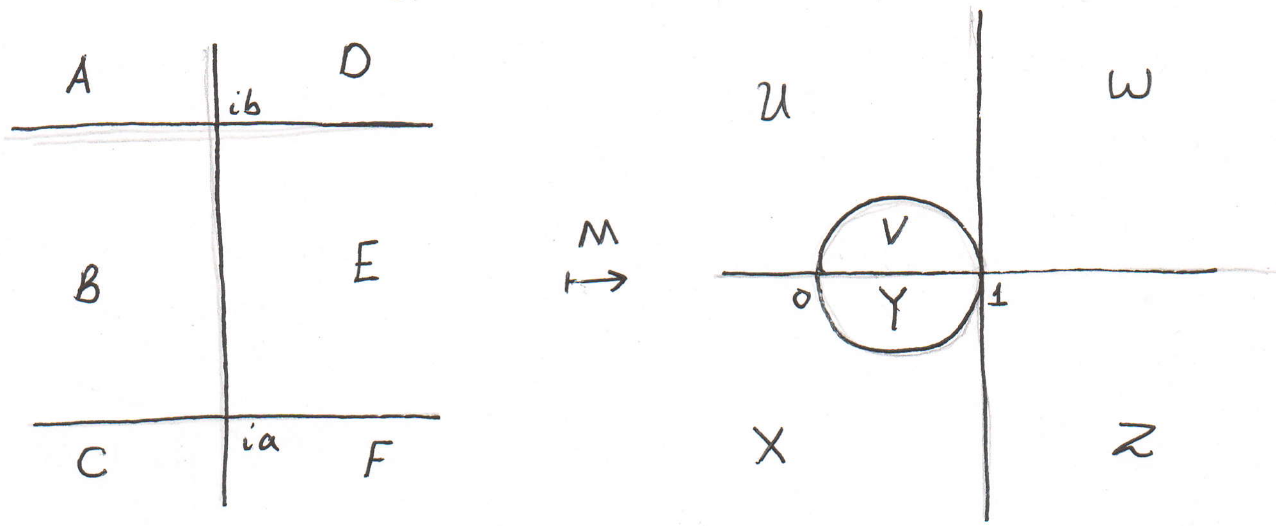


Fig. 1.

Fig. 2.

Solution: Observe first that $M(L_3) = \mathbb{R}_\infty$. This follows from tracking the 3 points $M(0) = \frac{a}{b} \in \mathbb{R}$, $M(\infty) = 1 \in \mathbb{R}$, and $M(ib) = \infty$. Since $ib \in L_1 \cap L_3$ and these lines are orthogonal, $M(L_1) = \{z : \operatorname{Re} z = 1\}$. Finally, $ib \notin L_2$ so $M(L_2)$ does not contain ∞. Hence $M(L_2)$ must be a circle. This circle contains the points $0 = M(ia)$ and $1 = M(\infty)$ and it has to be orthogonal to $M(L_3) = \mathbb{R}_\infty$ since $L_2 \perp L_3$. We use the orientation principle to identify the regions as follows:

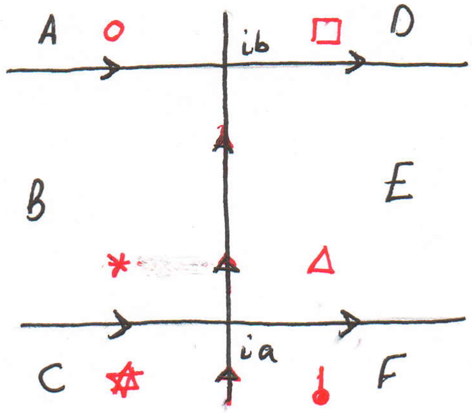


Fig. 1.

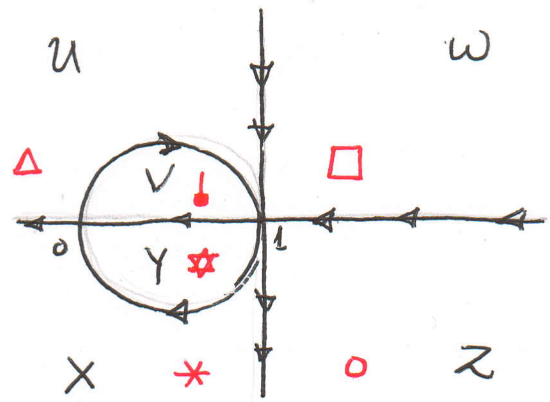


Fig. 2.

The orientation of L_3 is given by (ia, ib, ∞) and is carried to $(0, \infty, 1)$. Thus $M(L_3) = \mathbb{R}_\infty$ is oriented in the direction from right to left.

In a similar way, given the orientation $(\infty, ib, b-a+ib)$ of L_1, the corresponding orientation of $M(L_1)$ is $(1, \infty, 1+i)$ which is top to bottom.

Given the orientation $(\infty, ia, b-a+ia)$ of L_2 we obtain $(1, 0, \frac{1+i}{2})$ as the orientation of $M(L_2)$.

We can now assign the regions in fig. 1 to corresponding regions in Fig. 2. For instance, A is above L_1 and to the left of L_3. Hence, it must be below $M(L_3)$ and to the right of $M(L_1)$. Hence $M(A) = Z$. Similarly $M(B) = X$, $M(C) = Y$, $M(D) = W$, $M(E) = U$, and $M(F) = V$.